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Plasticity theory for fibre-reinforced composites

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Abstract. Recent experiments reported in [1] show that for a boron-aluminium fibre-reinforced composite plastic yielding is effectively independent of tension in the fibre direction over a wide range of values of this tension. This confirms a long-standing conjecture by the author and colleagues. The results have major implications for the formulation of plasticity theories for fibre-reinforced materials. This paper reviews the theory of anisotropic plasticity based on the usual assumptions of plasticity theory, together with the property that yielding is not affected by a superposed tension in the fibre direction. Yield conditions, flow rules and hardening rules are formulated for uniaxial reinforcement; brief consideration is given to a material reinforced by two families of fibres.

1. Introduction

Dvorak, Bahei-el-din, Macheret and Liu [1] have recently reported some important experiments on the elastic-plastic behaviour of a fibre-reinforced boron-aluminium composite. The experiments were performed on axially reinforced tubular specimens loaded by axial tension, torsion and internal pressure. Among other results, they show that over a wide range of values of the direct stress in the fibre-direction (essentially over the range in which the yield or failure stress of the fibre is not exceeded) the yield behaviour of the composite is independent of this direct stress in the fibre direction. The effect is shown in Fig. 1, which is based on Fig. 7 of [1]. The yield surface in the relevant stress space is 'sausage-shaped', being a fairly long cylinder with closed ends. It is also shown by the extensive experiments described in [1] that subsequent to plastic deformation the yield surface retains this sausage-shape but translates in stress-space, demonstrating a large degree of kinematic hardening.

These results provide striking confirmation of a conjecture made by the author and colleagues over twenty years ago [2–5]. The results have important implications for the formulation of theories of plasticity for fibre-reinforced composites, with particular reference to metal-matrix composites. These implications have been explored in previous publications [2–15] but prior to the availability of experimental information, the theory had to be regarded in a rather tentative manner. As the experimental data now exist, it seems worthwhile to draw attention to this work. This paper is a summary of the theory proposed in the papers cited above.

A theory of the *rigid-plastic* plastic behaviour of materials reinforced by a single family of fibres was formulated by Mulhern, Rogers and Spencer [2]. This theory assumed the material to be inextensible in the fibre direction, and it was shown that, as a consequence of this assumption and the associated flow rule, the yield function is independent of the fibre tension (that is, the direct stress in the fibre direction). Subsequently the same authors [3] formulated an *elastic-plastic* theory of fibre-reinforced materials, in which it was assumed that the *plastic* part of the extensional strain in the fibre direction is zero, and in this case also it follows from the associated flow rule that plastic yielding is independent of fibre

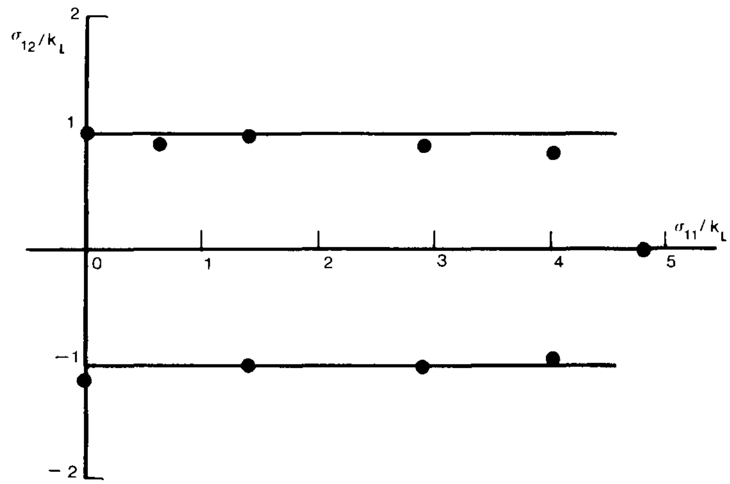

Fig. 1. Experimental results by Dvorak et al. [1] for initial yield of a boron-aluminium fibre-reinforced composite, with fibres parallel to the x_1-axis.

tension. However, this theory does *not* require inextensible fibres, because it admits an elastic strain in the fibre direction.

Smith and Spencer [4] formulated a *rigid-plastic* theory for material reinforced by two families of inextensible fibres, but adopted a slightly different point of view; they postulated that yielding is independent of fibre tensions. It then follows, if the associated flow rule is adopted, that the plastic extensional strain in a fibre direction is zero. The same viewpoint was taken in several subsequent publications, for example [5–15]. The assumptions of (a) inextensibility in the fibre direction, and (b) yield independent of fibre tension, are, in conjunction with the associated flow rule, virtually equivalent in rigid-plastic theory. However, in an elastic-plastic theory, (b) is less restrictive than (a), because it permits an axial elastic strain. Since (b) is susceptible to direct experimental observation, it seems to be the more natural choice of postulate for materials which exhibit this kind of behaviour.

2. Initial yield conditions – one family of fibres

We refer all vector and tensor quantities to a system of rectangular cartesian coordinates x_i. The Cauchy stress is denoted by $\boldsymbol{\sigma}$, with cartesian components σ_{ij}.

In most theories of metal plasticity, we postulate a yield function $f(\sigma_{ij})$ such that in admissible stress states $f \leq 0$, with $f = 0$ when plastic deformation is taking place. If the plastic material is isotropic then f can be expressed as a function of the stress invariants $\operatorname{tr} \boldsymbol{\sigma}$, $\operatorname{tr} \boldsymbol{\sigma}^2$ and $\operatorname{tr} \boldsymbol{\sigma}^3$. In *isotropic* metal plasticity it is observed experimentally that for many materials yielding is effectively independent of a superposed hydrostatic pressure, or equivalently of $\operatorname{tr} \boldsymbol{\sigma}$. This observation greatly simplifies the formulation and application of the theory. It is incorporated into the theory by restricting f to depend on $\boldsymbol{\sigma}$ only through the deviatoric stress $\mathbf{s}$, where

$$\mathbf{s} = \boldsymbol{\sigma} - \tfrac{1}{3}\mathbf{I} \operatorname{tr} \boldsymbol{\sigma} . \tag{2.1}$$

Then $\operatorname{tr} \mathbf{s} = 0$ and f can be expressed as a function of $\operatorname{tr} \mathbf{s}^2$ and $\operatorname{tr} \mathbf{s}^3$.

For anisotropic materials f is a function of $\boldsymbol{\sigma}$ (or s) which is invariant under the appropriate transformation group which describes the anisotropy. A material reinforced by a single family of fibres, randomly distributed in their cross-sectional planes, is locally transversely isotropic with respect to the local fibre direction, which can be characterised by a unit vector **a**. In this case f is a function of the five invariants [2, 5, 12]

$$\operatorname{tr}\boldsymbol{\sigma}, \quad \operatorname{tr}\boldsymbol{\sigma}^2, \quad \operatorname{tr}\boldsymbol{\sigma}^3, \quad \mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a}, \quad \mathbf{a}, \boldsymbol{\sigma}^2\cdot\mathbf{a}. \tag{2.2}$$

For fibre-reinforced metals we expect initial yielding to remain independent of tr $\boldsymbol{\sigma}$. We now have in [1] experimental confirmation that f is independent of the stress $\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a}$ in the fibre direction over a wide range of values of this stress. The extra-stress s′, defined as

$$\mathbf{s}' = \boldsymbol{\sigma} - \tfrac{1}{2}(\operatorname{tr}\boldsymbol{\sigma} - \mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a})\mathbf{I} + \tfrac{1}{2}(\operatorname{tr}\boldsymbol{\sigma} - 3\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a})\mathbf{a}\otimes\mathbf{a}, \tag{2.3}$$

(where $\mathbf{a}\otimes\mathbf{a}$ denotes the tensor product) has the property that s′ is unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma} - p\mathbf{I} + T\mathbf{a}\otimes\mathbf{a}$ for arbitrary p (representing a hydrostatic pressure) and T (representing a tension in the fibre direction). Thus s′ is independent of hydrostatic pressure and fibre tension. Consequently, if $f(\sigma_{ij})$ is expressed as a function of s′, then f is independent of hydrostatic pressure and fibre tension. Thus s′ has the same role in relation to independence of hydrostatic pressure and fibre tension as s has in relation to independence of hydrostatic pressure only. Furthermore, it follows from (2.3) that

$$\operatorname{tr}\mathbf{s}' = 0, \quad \mathbf{a}\cdot\mathbf{s}'\cdot\mathbf{a} = 0. \tag{2.4}$$

Now the set (2.2) is equivalent to

$$\operatorname{tr}\boldsymbol{\sigma}, \quad \operatorname{tr}\mathbf{s}'^2, \quad \operatorname{tr}\mathbf{s}'^3, \quad \mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a}, \quad \mathbf{a}\cdot\mathbf{s}'^2\cdot\mathbf{a},$$

so, if f is independent of tr $\boldsymbol{\sigma}$ and $\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a}$, and the material is transversely isotropic with respect to the direction **a**, then f may be expressed as a function of $\operatorname{tr}\mathbf{s}'^2$, $\mathbf{a}\cdot\mathbf{s}'^2\cdot\mathbf{a}$, $\operatorname{tr}\mathbf{s}'^3$. For our purposes it is more convenient to use the equivalent set

$$J_1 = \tfrac{1}{2}\operatorname{tr}\mathbf{s}'^2 - \mathbf{a}\cdot\mathbf{s}'^2\cdot\mathbf{a}, \quad J_2 = \mathbf{a}\cdot\mathbf{s}'^2\cdot\mathbf{a}, \quad J_3 = \tfrac{1}{3}\operatorname{tr}\mathbf{s}'^3. \tag{2.5}$$

The restriction to dependence on these three invariants, together with the relations (2.4), represents a considerable simplification compared to a (2.2).

In applications it is usually necessary to specify a particular form for f. In isotropic plasticity the commonly used yield functions are those which correspond to von Mises' and Tresca's yield conditions. For transversely isotropic materials, two forms which have been found useful in applications, and to agree well with experiment, are

$$\text{(a)} \quad f = \frac{1}{k_T^2}J_1 + \frac{1}{k_L^2}J_2 - 1, \tag{2.6}$$

and

$$\text{(b)} \quad f = \begin{cases} \dfrac{J_1^{1/2}}{k_T} - 1, & \text{for } J_2 \leq k_L^2, \\[2ex] \dfrac{J_2^{1/2}}{k_L} - 1, & \text{for } J_1 \leq k_T^2. \end{cases} \tag{2.7}$$

Here k_T and k_L are shear yield stresses for shear on planes containing the fibres, in directions transverse and parallel to the fibres respectively. The form (2.6) is the most general yield function of the required form that is quadratic in the stress components, and so may be regarded as an analogue of von Mises' condition. The form (2.7) is a maximum shear stress condition, and so is analogous to Tresca's condition for isotropic plasticity.

If the coordinate system is chosen so that **a** lies parallel to the x_1-axis (in conformity with the notation used in [1]), then $\mathbf{a} = (1, 0, 0)$ and (2.6) becomes

$$f = \frac{1}{k_T^2}\left\{\frac{1}{4}(\sigma_{22} - \sigma_{33})^2 + \sigma_{23}^2\right\} + \frac{1}{k_L^2}(\sigma_{12}^2 + \sigma_{13}^2) - 1\,, \tag{2.8}$$

whilst (2.7) takes the form

$$f = \begin{cases} \dfrac{1}{k_T}\left\{\dfrac{1}{4}(\sigma_{22} - \sigma_{33})^2 + \sigma_{23}^2\right\}^{1/2} - 1\,, & \text{for } (\sigma_{12}^2 + \sigma_{13}^2)^2 \leq k_L^2\,, \\ \dfrac{1}{k_L}(\sigma_{12}^2 + \sigma_{13}^2)^{1/2} - 1\,, & \text{for } \dfrac{1}{4}(\sigma_{22} - \sigma_{33})^2 + \sigma_{23}^2 \leq k_T^2\,. \end{cases} \tag{2.9}$$

The parameters k_T and k_L are identified as shear yield stresses for shear on planes containing the fibres, in directions transverse and parallel to the fibres respectively.

3. Associated flow rule

We denote the velocity vector by **v** and its cartesian components by v_i. The rate of deformation tensor **d** has components d_{ij} defined by

$$d_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right). \tag{3.1}$$

As is usual in plasticity, 'time' here need not be real time, but may be any parameter that orders the sequence of events. In the formulation of isotropic elastic-plastic theory it is usual to decompose **d** into an elastic part $\mathbf{d}^e$ and a plastic part $\mathbf{d}^p$. The decomposition may be done in various ways. When the elastic part of the strain is small there is little difference between the various formulations so we adopt a simple additive decomposition and set

$$\mathbf{d} = \mathbf{d}^e + \mathbf{d}^p\,. \tag{3.2}$$

The elastic strain-rate $\mathbf{d}^e$ is assumed to be related to the stress-rate by the elastic stress-strain law. For large deformations the stress-rate must be defined in a properly objective manner. Anisotropic behaviour does not raise any new issues in this regard so we do not elaborate on this subject. We observe that linear elastic stress-strain relations for transversely isotropic materials are well-known [e.g. 5, 9, 11–15].

It is usual to assume that the plastic strain-rate is related to the stress by the associated flow rule

$$d_{ij}^p = \dot{\lambda}\,\frac{\partial f}{\partial \sigma_{ij}}\,, \tag{3.3}$$

where for a perfectly plastic material $\dot{\lambda}$ is a positive scalar factor of proportionality, and for a strain-hardening plastic material $\dot{\lambda}$ depends on strain history. We consider the hardening case further below. The arguments used in support of the adoption of (3.3) are not affected by anisotropy or kinematic constraints. We follow common practice and assume (3.3). Then when f is expressed as a function of J_1, J_2 and J_3 we have

$$d_{ij}^p = \dot{\lambda} \sum_{\alpha=1}^{3} \frac{\partial f}{\partial J_\alpha} \frac{\partial J_\alpha}{\partial \sigma_{ij}} .$$

It is straightforward to verify that

$$\frac{\partial J_\alpha}{\partial \sigma_{ii}} = 0 , \qquad a_i a_j \frac{\partial J_\alpha}{\partial \sigma_{ij}} = 0 , \quad (\alpha = 1, 2, 3) .$$

Hence, in this model

$$\operatorname{tr} \mathbf{d}^p = 0 , \qquad \mathbf{a} \cdot \mathbf{d}^p \cdot \mathbf{a} = 0 . \tag{3.4}$$

Therefore, plastic incompressibility and plastic inextensibility in the fibre direction are *consequences* of the assumption that yield is independent of hydrostatic pressure and fibre tension, together with the associated flow rule.

For the particular forms of f given by (2.6) and (2.7), (3.3) gives, for the yield function (2.6)

$$\mathbf{d}^p = \dot{\lambda} \left\{ \frac{1}{k_T^2} (\mathbf{s}' - \mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' - \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a}) + \frac{1}{k_L^2} (\mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' + \mathbf{s}' \mathbf{a} \otimes \mathbf{a}) \right\} , \tag{3.5}$$

and for the yield function (2.7)

$$\mathbf{d}^p = \begin{cases} \dot{\mu}(\mathbf{s}' - \mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' - \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a}) , & J_1 = k_T^2 , \quad J_2 < k_L^2 , \\ \dot{\nu}(\mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' + \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a}) , & J_1 < k_T^2 , \quad J_2 = k_L^2 , \\ \dot{\mu} \mathbf{s}' + (\dot{\nu} - \dot{\mu})(\mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' + \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a}) , & J_1 = k_T^2 , \quad J_2 = k_L^2 , \end{cases} \tag{3.6}$$

where $\dot{\mu}$ and $\dot{\nu}$ are positive multipliers.

If the coordinate system is chosen so that $\mathbf{a} = (1, 0, 0)$, then from (2.3)

$$\mathbf{s}' - \mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' - \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(\sigma_{22} - \sigma_{33}) & \sigma_{23} \\ 0 & \sigma_{23} & \frac{1}{2}(\sigma_{33} - \sigma_{22}) \end{bmatrix} ,$$

$$\mathbf{a} \otimes \mathbf{a} \cdot \mathbf{s}' + \mathbf{s}' \cdot \mathbf{a} \otimes \mathbf{a} = \begin{bmatrix} 0 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & 0 & 0 \\ \sigma_{13} & 0 & 0 \end{bmatrix} ,$$

and the resulting expressions for $\mathbf{d}^p$ are given by inserting these in (3.5) and (3.6).

4. Proportional hardening

The post-yield behaviour even of initially isotropic plastic materials is complicated. An idealisation often introduced in isotropic plasticity theory is that the current state of hardening can be described by a single parameter, usually taken to be the shear yield stress, k. It is further assumed that the current value of k depends on the deformation history through an 'equivalent strain', ε, such that the convected derivative $\dot{\varepsilon}$ is a function of the current plastic strain rate d^p_{ij} and homogeneous of degree -1 in time. It is usually not explicitly stated, but is clearly necessary, that for isotropic response $\dot{\varepsilon}$ must be an invariant of d^p_{ij}. Thus at most $\dot{\varepsilon}$ is a function of

$$d^p_{ii}\,, \qquad d^p_{ij}d^p_{ij}\,, \qquad d^p_{ij}d^p_{jk}d^p_{ki}\,. \tag{4.1}$$

However $d^p_{ii} = 0$, and so is discarded. The invariable practice is to identify (apart from a numerical factor which is unimportant)

$$\dot{\varepsilon} = (d^p_{ij}d^p_{ij})^{1/2}\,, \tag{4.2}$$

although there seems no essential reason not to include also the third of (4.1). From (4.2) we then have

$$\varepsilon = \int_0^t (d^p_{ij}d^p_{ij})^{1/2}\,\mathrm{d}t\,,$$

and $k = k(\varepsilon)$. This constitutive assumption is termed isotropic hardening.

We seek to extend these ideas to fibre-reinforced plastic materials. It is clear that in this case the description of the current state of hardening will require more than a single parameter; as a minimum the two shear yield stresses k_T and k_L are needed. Nor can it be assumed that k_T and k_L will depend on the deformation history through a single parameter, because, for example, shearing across and along the fibres may well affect the hardening differently. Thus we have to define several 'equivalent strain' parameters whose time derivatives are invariant under the symmetry transformation appropriate to transverse isotropy. These are the invariants of $\mathbf{d}^p$ analogous to the invariants (2.2) of $\boldsymbol{\sigma}$, namely

$$\operatorname{tr}\mathbf{d}^p\,, \qquad \operatorname{tr}(\mathbf{d}^p)^2\,, \qquad \operatorname{tr}(\mathbf{d}^p)^3\,, \qquad \mathbf{a}\cdot\mathbf{d}^p\cdot\mathbf{a}\,, \qquad \mathbf{a}\cdot(\mathbf{d}^p)^2\cdot\mathbf{a}\,. \tag{4.3}$$

However $\operatorname{tr}\mathbf{d}^p$ and $\mathbf{a}\cdot\mathbf{d}^p\cdot\mathbf{a}$ are both zero. We follow the practice of isotropic plasticity and discard the third of (4.3), leaving two equivalent strain parameters which are derived from $\operatorname{tr}(\mathbf{d}^p)^2$ and $\mathbf{a}\cdot(\mathbf{d}^p)^2\cdot\mathbf{a}$. A convenient choice is ε_T and ε_L, where

$$\dot{\varepsilon}_T^2 = \tfrac{1}{2}\operatorname{tr}(\mathbf{d}^p)^2 - \mathbf{a}\cdot(\mathbf{d}^p)^2\cdot\mathbf{a}\,, \qquad \dot{\varepsilon}_L^2 = \mathbf{a}\cdot(\mathbf{d}^p)^2\cdot\mathbf{a}\,. \tag{4.4}$$

In particular, if the coordinate system is chosen so that $\mathbf{a} = (1,0,0)$, then

$$\dot{\varepsilon}_T^2 = \tfrac{1}{4}(d_{22} - d_{33})^2 + d_{23}^2\,, \qquad \dot{\varepsilon}_L^2 = d_{12}^2 + d_{13}^2\,,$$

and it is clear that ε_T and ε_L are associated with transverse and longitudinal shearing respectively. Our constitutive assumption is then

$$k_T = k_T(\varepsilon_T, \varepsilon_L)\,, \qquad k_L = k_L(\varepsilon_T, \varepsilon_L)\,. \tag{4.5}$$

This appears to be the simplest analogue for transverse isotropy of the isotropic hardening theory of isotropic plasticity. Since it seems inconsistent to talk of isotropic hardening of an anisotropic material, we use the term 'proportional hardening' for this theory. Under proportional hardening the yield surface has a fixed centre at the origin of stress space and expands (for strain-hardening material) as plastic deformation proceeds. The expansion is uniform in any given direction in stress space, but differs with the direction. Thus for example a yield surface that was initially a hyper-ellipse would evolve to another hyper-ellipse, but in general the ratios of the lengths of the principal axes would not remain constant.

It remains to relate $\dot{\lambda}$ to the loading parameters. From (4.5)

$$\dot{k}_T = \frac{\partial k_T}{\partial \varepsilon_T}\dot{\varepsilon}_T + \frac{\partial k_T}{\partial \varepsilon_L}\dot{\varepsilon}_L\,, \qquad \dot{k}_L = \frac{\partial k_L}{\partial \varepsilon_T}\dot{\varepsilon}_T + \frac{\partial k_L}{\partial \varepsilon_L}\dot{\varepsilon}_L\,. \tag{4.6}$$

We confine discussion to the yield functions (2.6) and (2.7) and their associated flow rules (3.5) and (3.6). For (2.6) we have, using (3.5) and (4.4)

$$\dot{\varepsilon}_T^2 = \dot{\lambda}^2 k_T^{-4} J_1\,, \qquad \dot{\varepsilon}_L^2 = \dot{\lambda}^2 k_L^{-4} J_2\,. \tag{4.7}$$

Hence from (2.6), when $f = 0$

$$\dot{\lambda}^2 = k_T^2 \dot{\varepsilon}_T^2 + k_L^2 \dot{\varepsilon}_L^2\,. \tag{4.8}$$

Also from (2.6), during plastic loading, when $\dot{f} = 0$ we have

$$k_T^{-2}\dot{J}_1 + k_L^{-2}\dot{J}_2 - 2(k_T^{-3}\dot{k}_T J_1 + k_L^{-3}\dot{k}_L J_2) = 0\,. \tag{4.9}$$

Hence, from (4.6), (4.8) and (4.9)

$$\dot{\lambda}\left\{J_1^{1/2}k_T^{-2}\left(k_T^{-3}J_1\frac{\partial k_T}{\partial \varepsilon_T} + k_L^{-3}J_2\frac{\partial k_L}{\partial \varepsilon_T}\right) + J_2^{1/2}k_L^{-2}\left(k_T^{-3}J_1\frac{\partial k_T}{\partial \varepsilon_L} + k_L^{-3}J_2\frac{\partial k_L}{\partial \varepsilon_L}\right)\right\}$$
$$= \frac{1}{2}(k_T^{-2}\dot{J}_1 + k_L^{-2}\dot{J}_2)\,. \tag{4.10}$$

This determines $\dot{\lambda}$ in terms of the current stress, the current values of k_T and k_L, the hardening parameters $\partial k_T/\partial \varepsilon_T$, $\partial k_T/\partial \varepsilon_L$, $\partial k_L/\partial \varepsilon_T$, $\partial k_L/\partial \varepsilon_L$, and the rate of change of the stress invariants J_1 and J_2. The corresponding deformation-rate, in plastic loading, is then given by (3.5).

The plastic work-rate $\dot{W}_p$ is given by

$$\dot{W}_p = d_{ij}^p \sigma_{ij} = d_{ij}^p s'_{ij}\,, \tag{4.11}$$

and hence, from (2.5) and (3.5), for the yield function (2.6)

$$\dot{W}_p = 2\dot{\lambda}\left(\frac{J_1}{k_T^2} + \frac{J_2}{k_L^2}\right) = 2\dot{\lambda}\,, \tag{4.12}$$

and, from (4.8)

$$\dot{W}_p = 2(k_T^2\dot{\varepsilon}_T^2 + k_L^2\dot{\varepsilon}_L^2)^{1/2}. \tag{4.13}$$

Some possibilities for simplification suggest themselves. One is that k_T and k_L are functions only of the plastic work W_p. This is a plausible assumption in the case of a metal matrix reinforced by elastic fibres, if it is considered that the hardening of the composite originates in isotropic hardening of the matrix. In this case, from (4.12)

$$\dot{k}_T = 2\,\frac{\mathrm{d}k_T}{\mathrm{d}W_p}\,\dot{\lambda}\,, \qquad \dot{k}_L = 2\,\frac{\mathrm{d}k_L}{\mathrm{d}W_p}\,\dot{\lambda}\,, \tag{4.14}$$

and it follows from (4.9) and (4.14) that

$$\dot{\lambda}\left\{k_T^{-3}J_1\,\frac{\mathrm{d}k_T}{\mathrm{d}W_p} + k_L^{-3}J_2\,\frac{\mathrm{d}k_L}{\mathrm{d}W_p}\right\} = \frac{1}{4}\,(k_T^{-2}\dot{J}_1 + k_L^{-2}\dot{J}_2)\,, \tag{4.15}$$

which determines $\dot{\lambda}$ in terms of the current stress and stress rate. A further possible simplification is to suppose that the ratio k_T/k_L remains constant, which is also plausible if the composite hardening results solely from isotropic hardening of the matrix.

Another possible simplification arises if it is assumed that the mechanisms for strain-hardening in shear in the directions parallel and normal to the fibres are independent. In this case

$$k_T = k_T(\varepsilon_T)\,, \qquad k_L = k_L(\varepsilon_L)\,,$$

and (4.10) simplifies accordingly.

Similar considerations apply if the yield function (2.7) is adopted, with its associated flow rule (3.6). In this case, from (4.4) and (3.6)

$$\dot{\varepsilon}_T^2 = \begin{cases}\dot{\mu}^2 J_1 \\ 0 \\ \dot{\mu}^2 J_1\end{cases}, \qquad \dot{\varepsilon}_L^2 = \begin{cases}0 & J_1 = k_T^2\,, \quad J_2 < k_L^2 \\ \dot{\nu}^2 J_2\,, & J_1 < k_T^2\,, \quad J_2 = k_L^2 \\ \dot{\nu}^2 J_2 & J_1 = k_T^2\,, \quad J_2 = k_L^2\end{cases}. \tag{4.16}$$

During plastic loading, when $\dot{f} = 0$, either

$$J_1 = k_T^2\,, \qquad \dot{J}_1 = 2k_T\dot{k}_T\,, \qquad J_2 < k_L^2\,,$$

or

$$J_1 < k_T^2\,, \qquad J_2 = k_L^2\,, \qquad \dot{J}_2 = 2k_L\dot{k}_L\,, \tag{4.17}$$

or

$$J_1 = k_T^2\,, \qquad J_2 = k_L^2\,, \qquad \dot{J}_1 = 2k_T\dot{k}_T\,, \qquad \dot{J}_2 = 2k_L\dot{k}_L\,.$$

Hence from (4.6), (4.16) and (4.17), during plastic loading

$$\begin{aligned}
&\dot{\mu}\,\frac{\partial k_T}{\partial \varepsilon_T} = \frac{1}{2}\,k_T^{-2}\dot{J}_1 \quad \text{if } J_1 = k_T^2\,, \quad J_2 < k_L^2\,, \\
&\dot{\nu}\,\frac{\partial k_L}{\partial \varepsilon_L} = \frac{1}{2}\,k_L^{-2}\dot{J}_2 \quad \text{if } J_1 < k_T^2\,, \quad J_2 = k_L^2\,,
\end{aligned} \tag{4.18}$$

and at a vertex $J_1 = k_T^2$, $J_2 = k_L^2$ of the yield surface, $\dot{\mu}$ and $\dot{\nu}$ are given by

$$\begin{aligned} \dot{\mu} k_T \frac{\partial k_T}{\partial \varepsilon_T} + \dot{\nu} k_L \frac{\partial k_T}{\partial \varepsilon_L} &= \frac{1}{2} k_T^{-1} \dot{J}_1 , \\ \dot{\mu} k_T \frac{\partial k_L}{\partial \varepsilon_T} + \dot{\nu} k_L \frac{\partial k_L}{\partial \varepsilon_L} &= \frac{1}{2} k_L^{-1} \dot{J}_2 . \end{aligned} \tag{4.19}$$

The plastic work-rate associated with (3.6) is, using (4.16)

$$\dot{W}_p = \begin{cases} 2\dot{\mu} J_1 = 2\dot{\mu} k_T^2 = 2\dot{\varepsilon}_T k_T & \text{when } J_1 = k_T^2 , \quad J_2 < k_L^2 \\ 2\dot{\nu} J_2 = 2\dot{\nu} k_L^2 = 2\dot{\varepsilon}_L k_L & \text{when } J_1 < k_T^2 , \quad J_2 = k_L^2 . \\ 2(\dot{\mu} J_1 + \dot{\nu} J_2) = 2(\dot{\mu} k_T^2 + \dot{\nu} k_L^2) = 2(\dot{\varepsilon}_T k_T + \dot{\varepsilon}_L k_L) & \text{when } J_1 = k_T^2 , \quad J_2 = k_L^2 \end{cases} \tag{4.20}$$

In the special case in which k_T and k_L depend only on W_p, it follows from (4.20) that

$$\dot{k}_T = 2 \frac{\mathrm{d}k_T}{\mathrm{d}W_p} \begin{cases} \dot{\mu} k_T^2 , \\ \dot{\nu} k_L^2 , \\ \dot{\mu} k_T^2 + \dot{\nu} k_L^2 , \end{cases} \qquad \dot{k}_L = 2 \frac{\mathrm{d}k_L}{\mathrm{d}W_p} \begin{cases} \dot{\mu} k_T^2 , & J_1 = k_T^2 , \quad J_2 < k_L^2 \\ \dot{\nu} k_L^2 , & J_1 < k_T^2 , \quad J_2 = k_L^2 . \\ \dot{\mu} k_T^2 + \dot{\nu} k_L^2 , & J_1 = k_T^2 , \quad J_2 = k_L^2 \end{cases} \tag{4.21}$$

Then, from (4.17) and (4.21), $\dot{\mu}$ and $\dot{\nu}$ are given by

$$\dot{\mu} \frac{\mathrm{d}k_T}{\mathrm{d}W_p} = \frac{1}{4} k_T^{-3} \dot{J}_1 , \quad \text{when } J_1 = k_T^2 , \quad J_2 < k_L^2 ,$$

$$\dot{\nu} \frac{\mathrm{d}k_L}{\mathrm{d}W_p} = \frac{1}{4} k_L^{-3} \dot{J}_2 , \quad \text{when } J_1 < k_T^2 , \quad J_2 = k_L^2 ,$$

$$\left. \begin{aligned} (\dot{\mu} k_T^2 + \dot{\nu} k_L^2) \frac{\mathrm{d}k_T}{\mathrm{d}W_p} &= \frac{1}{4} \dot{J}_1 k_T^{-1} \\ (\dot{\mu} k_T^2 + \dot{\nu} k_L^2) \frac{\mathrm{d}k_L}{\mathrm{d}W_p} &= \frac{1}{4} \dot{J}_2 k_L^{-1} \end{aligned} \right\} \quad \text{when } J_1 = k_T^2 , \quad J_2 = k_L^2 .$$

Except perhaps when a vertex of the yield surface is involved, the 'Tresca-type' yield function (2.7) usually leads to simpler formulae than the 'von Mises-type' yield function (2.6).

5. Kinematic hardening

Kinematic hardening is characterised by a tensor $\boldsymbol{\alpha}$ termed the 'back stress' or 'shift tensor' which represents a translation of the yield surface in stress space that depends on the strain history. For materials of the class under consideration, in the yield condition $\mathbf{s}'$ is replaced by $\mathbf{s}' - \boldsymbol{\alpha}$. The tensor $\boldsymbol{\alpha}$ must satisfy the same constraints as $\mathbf{s}'$, so we require

$$\operatorname{tr} \boldsymbol{\alpha} = 0 , \qquad \mathbf{a} \cdot \boldsymbol{\alpha} \cdot \mathbf{a} = 0 . \tag{5.1}$$

It is necessary to specify evolution equations for $\boldsymbol{\alpha}$. For an isotropic material, it is common to adopt the rule proposed by Prager

$$\dot{\boldsymbol{\alpha}} = c(\varepsilon)\mathbf{d}^p . \tag{5.2}$$

For finite deformations, $\dot{\boldsymbol{\alpha}}$ must be an objective tensor-rate; the appropriate choice of $\dot{\boldsymbol{\alpha}}$ is a matter of discussion which we do not enter into here.

We seek a suitable generalisation of (5.2) for transversely isotropic materials. It seems clear that $\dot{\boldsymbol{\alpha}}$ must depend on both ε_L and ε_T. Also the natural generalisation of (5.2) is to propose that $\dot{\boldsymbol{\alpha}}$ is a linear tensor function of $\mathbf{d}^p$ with the appropriate invariance properties. This leads us to the form

$$\dot{\boldsymbol{\alpha}} = c_1(\varepsilon_T, \varepsilon_L)(\mathbf{d}^p - \mathbf{a}\otimes\mathbf{a}\cdot\mathbf{d}^p - \mathbf{d}^p\cdot\mathbf{a}\otimes\mathbf{a}) + c_2(\varepsilon_T, \varepsilon_L)(\mathbf{a}\otimes\mathbf{a}\cdot\mathbf{d}^p + \mathbf{d}^p\cdot\mathbf{a}\otimes\mathbf{a}) , \tag{5.3}$$

which is proposed as the appropriate form for fibre-reinforced plastic materials.

If the coordinate system is such that $\mathbf{a} = (1, 0, 0)$ then

$$\alpha_{11} = 0 , \qquad \alpha_{22} + \alpha_{33} = 0$$

and (5.3) becomes

$$\begin{bmatrix} 0 & \dot{\alpha}_{12} & \dot{\alpha}_{13} \\ \dot{\alpha}_{12} & \frac{1}{2}(\dot{\alpha}_{22} - \dot{\alpha}_{33}) & \dot{\alpha}_{23} \\ \dot{\alpha}_{13} & \dot{\alpha}_{23} & \frac{1}{2}(\dot{\alpha}_{33} - \dot{\alpha}_{22}) \end{bmatrix} = c_1(\varepsilon_T, \varepsilon_L) \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(d^p_{22} - d^p_{33}) & d^p_{23} \\ 0 & d^p_{23} & \frac{1}{2}(d^p_{33} - d^p_{22}) \end{bmatrix}$$
$$+ c_2(\varepsilon_T, \varepsilon_L) \begin{bmatrix} 0 & d^p_{12} & d^p_{13} \\ d^p_{12} & 0 & 0 \\ d^p_{13} & 0 & 0 \end{bmatrix}$$

6. General transverse isotropy

In the theory of Sections 2–5 it was assumed that yielding is independent of the fibre tension, and this led to the result $\mathbf{a}\cdot\mathbf{d}^p\cdot\mathbf{a} = 0$. This assumption seems appropriate, and is supported by the results of [1], within the range of axial stress for which the yield or failure strain in the fibre is not exceeded. For brittle-elastic fibres, the theory applies up to failure by fibre breakage. For elastic-plastic fibres, an additional plastic flow regime involving plastic flow in the fibre direction must be included; for practical fibre composites, this will only become operative under fibre stress of large magnitude. Such an effect is incorporated in the bi-modal theory proposed by Dvorak and Bahei-el-din [16]. Extensions of the present theory to allow yield in the fibre direction have been outlined by Spencer [11, 12] and developed in more detail by Rogers [13, 14].

It is also demonstrated in [1] that fibre-reinforced materials show strong kinematic hardening, especially in relation to the stress in the fibre direction. This is due to residual stress in the fibre and matrix following plastic flow in the fibre direction. Brief discussions are given in [11], [13] and [14].

7. Reinforcement by two families of fibres

Similar considerations to those of Sections 2–5 can be applied in the case of an elastic-plastic material reinforced by two families of fibres (as for example, in a laminate comprised of many alternate layers of uniaxially reinforced material with alternating orientations). Such a material is, macroscopically, monoclinic in symmetry, or orthotropic if either (i) the two fibre directions are orthogonal or (ii) the two fibre directions are mechanically equivalent.

The two fibre directions are defined by unit vectors **a** and **b**, which are regarded as continuous vector fields. We denote by 2φ the angle between the two fibre directions, so that

$$\mathbf{a} \cdot \mathbf{b} = \cos 2\varphi \ . \tag{7.1}$$

The tensor

$$\begin{aligned} \mathbf{s}'' = \boldsymbol{\sigma} + (1 + 3\cos^2 2\varphi)^{-1}[&\{\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a} + \mathbf{b}\cdot\boldsymbol{\sigma}\cdot\mathbf{b} - (1+\cos^2 2\varphi)\operatorname{tr}\boldsymbol{\sigma}\}\mathbf{I} \\ &+ \{\operatorname{tr}\boldsymbol{\sigma} - (2\operatorname{cosec}^2 2\varphi)\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a} - (\operatorname{cosec}^2 2\varphi - 3\cot^2 2\varphi)\mathbf{b}\cdot\boldsymbol{\sigma}\cdot\mathbf{b}\}\mathbf{a}\otimes\mathbf{a} \\ &+ \{\operatorname{tr}\boldsymbol{\sigma} - (2\operatorname{cosec}^2 2\varphi)\mathbf{b}\cdot\boldsymbol{\sigma}\cdot\mathbf{b} - (\operatorname{cosec}^2 2\varphi - 3\cot^2 2\varphi)\mathbf{a}\cdot\boldsymbol{\sigma}\cdot\mathbf{a}\}\mathbf{b}\otimes\mathbf{b}] \end{aligned} \tag{7.2}$$

has the property that $\mathbf{s}''$ is independent of superposed hydrostatic pressure, of superposed fibre tension in the **a** direction, and of superposed fibre tension in the **b** direction, with

$$\operatorname{tr}\mathbf{s}'' = 0\ , \qquad \mathbf{a}\cdot\mathbf{s}''\cdot\mathbf{a} = 0\ , \qquad \mathbf{b}\cdot\mathbf{s}''\cdot\mathbf{b} = 0\ . \tag{7.3}$$

If yielding is independent of hydrostatic pressure and the two fibre tensions, the yield function f is an isotropic invariant of $\mathbf{s}''$, $\mathbf{a}\otimes\mathbf{a}$ and $\mathbf{b}\otimes\mathbf{b}$. Taking into account (7.3) it follows [4] that f can be expressed as a function of

$$\begin{aligned} &J_1 = \tfrac{1}{2}\operatorname{tr}\mathbf{s}''^2 - \mathbf{a}\cdot\mathbf{s}''^2\cdot\mathbf{a}\ , \qquad J_2 = \mathbf{a}\cdot\mathbf{s}''^2\cdot\mathbf{a}\ , \qquad J_3 = \tfrac{1}{3}\operatorname{tr}\mathbf{s}''^3\ , \\ &J_4 = \tfrac{1}{2}\operatorname{tr}\mathbf{s}''^2 - \mathbf{b}\cdot\mathbf{s}''^2\cdot\mathbf{b}\ , \qquad J_5 = \mathbf{a}\cdot\mathbf{s}''\cdot\mathbf{b}\cos 2\varphi\ , \qquad J_6 = \mathbf{a}\cdot\mathbf{s}''^2\cdot\mathbf{b}\cos 2\varphi \end{aligned} \tag{7.4}$$

and $\cos^2 2\varphi$. In the case when the two families of fibres are mechanically equivalent, then f has to be a symmetric function of **a** and **b**.

The most general *quadratic* yield function that satisfies these conditions is

$$f = \frac{J_1 + J_4}{c_1^2} + \frac{J_1 + 2J_2 - J_4}{c_2^2} + \frac{J_6}{c_3^2} - 1\ , \tag{7.5}$$

where c_1, c_2 and c_3 have dimensions of stress and are functions of $\cos 2\varphi$. If we choose the coordinate system so that the fibres lie in the planes $x_3 = \text{constant}$ and make angles $\pm\varphi$ with the x_1 axis, so that

$$\mathbf{a} = (\cos\varphi, \sin\varphi, 0)\ , \qquad \mathbf{b} = (\cos\varphi, -\sin\varphi, 0) \tag{7.6}$$

then (7.5) can be written in the form

$$f = \frac{1}{Y^2}\{(\sigma_{11} - \sigma_{33})\sin^2\varphi - (\sigma_{22} - \sigma_{33})\cos^2\varphi\}^2 + \frac{1}{k_1^2}\sigma_{13}^2 + \frac{1}{k_2^2}\sigma_{23}^2 - 1\ , \tag{7.7}$$

where Y, k_1 and k_2 can be related to c_1, c_2 and c_3. If we further denote

$$Y = Y_1 \sin^2\varphi = Y_2 \cos^2\varphi = Y_3|\cos^2\varphi - \sin^2\varphi| , \tag{7.8}$$

then k_1 and k_2 can be interpreted as shear yield stresses for shear on surfaces x_3 = constant in the x_1 and x_2 directions respectively, and Y_1, Y_2 and Y_3 as tensile yield stresses in the x_1, x_2 and x_3 directions respectively. The flow rule associated with (7.5) and (7.7) is given in [4] and [5]. Proportional and kinematic hardening rules can be formulated in a manner similar to that used in Sections 4–5, but we omit details.

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